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AN INVESTIGATION OF THE STABILITY AND NATURAL FREQUENCIES OF TRANSVERSE VIBRATION OF A FIXED-FIXED SANDWICH BEAM UNDER AXIAL LOADS

A THESIS

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AN INVESTIGATION OF THE STABILITY AND NATURAL FREQUENCIES

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SUMMARY

In this study, the natural frequencies and critical buckling loads of a fixed-fixed sandwich beam with axial loads were investigated using the method of Lagrangian multipliers.

Sandwich beams have been investigated in some detail by the Forest Products Laboratory and by Kansas State University. The use of the Lagrangian multiplier method as applied to fixed-fixed sandwich beams to determine natural frequencies of vibration was introduced in a work by M. E. Raville, En-Shiuh Ueng, and Ming-Min Lei of Kansas State University published in the Journal of Applied Mechanics of the ASME, in 1961.

In the present work, expressions for the desired natural frequencies and critical buckling loads are derived for a fixed-fixed sandwich beam by minimizing the total energy expressions in the beam with respect to the undetermined coefficients in the admissible set of orthogonal configuration functions. The total energy integral consists of the elastic energy of the vibrating beam, and the terms containing the constraint conditions placed on the system to meet boundary conditions. In determining the elastic energy of the beam, the facings are assumed to be thin plates, the modulus of elasticity of the core in the direction perpendicular to the facings is assumed infinite, and rotatory inertia is neglected. An infinite cosine series is assumed to represent the deflection, so that the boundary conditions involving the end slopes are identically satisfied term by term. The boundary conditions involving end deflections are satisfied by the series as a whole by properly relating

the coefficients of the series. The resulting equations for the determination of the eigenvalues consist of one infinite series using odd terms and one infinite series using even terms. The series containing even terms pertains to the odd modes of vibration and the series containing odd terms pertains to the even modes of vibration.

A particular example is chosen to illustrate the application of the derived equations and the first two modes of vibration and critical buckling loads are determined for this particular beam using iteration procedures on a digital computer. The beam chosen is 96 inches long with aluminum facings and an aluminum core. Six terms of each of the series are used, giving an accuracy for the results that is judged to be better than one and one-half percent.

The expressions derived in this study for the combined natural frequency and axial load of a sandwich beam are generalizations of the expressions derived in previous work for natural frequency of vibration alone. The numerical results obtained when the derived expressions are applied to a particular beam reflect the fact that the frequency is squared in the terms of the series, while the load is of first degree, hence the nonlinear nature of the curve of load versus natural frequency.

Further work is recommended in verifying the results of this problem experimentally. It is also recommended that the area in which homogeneous beam theory can be used as an approximation in the analysis of sandwich beams be more clearly defined by further investigation.

SYMBOLS

x, y, z	rectangular coordinates
a	length of beam
f	facing thickness
c	core thickness
P	axial load
u	deflection of the beam in the x direction
v	deflection of the beam in the y direction
u_c	deflection of the core in the x direction
v_c	deflection of the core in the y direction
A_o, A_m, B_m, C_m, A_i, B_i, C_i	configuration parameters
ω	natural frequency of vibration of the beam
t	time
τ_{xy}	shear stress in the core
G	modulus of rigidity of the core
σ_y	normal stress in the core
V_c	shear energy in the core
E_c	modulus of elasticity of the core
u_f	deflection of the facings in the x direction
v_f	deflection of the facings in the y direction
ϵ_{xMb}	membrane strains in the bottom facing in the x direction
ϵ_{xMt}	membrane strains in the top facing in the x direction

V_{Mb}	membrane strain energy in the bottom facing
V_{Mt}	membrane strain energy in the top facing
ν	Poisson's ratio
V_{Bft}	bending strain energy in the top facing
V_{Bfb}	bending strain energy in the bottom facing
ϵ_{xBt}	strain due to bending of the top facing
ϵ_{xBb}	strain due to bending of the bottom facing
V	total elastic strain energy
T_P	energy lost by the applied axial load
T_K	kinetic energy of the vibrating beam
ρ	mass density of the composite beam per unit length
V_{max}	maximum value of total elastic strain energy
T_{Pmax}	maximum value of energy lost by the applied load
T_{Kmax}	maximum value of the kinetic energy of the vibrating beam
U	modified energy
λ_1, λ_2	Lagrangian multipliers

CHAPTER I

INTRODUCTION

In this study, the natural frequencies of transverse vibration of a fixed-fixed sandwich beam under axial loading and the critical loads under which the beam will buckle are investigated using the method of Lagrangian multipliers.

Sandwich panels are made by bonding relatively thin, hard, strong, rigid facings to thick, lightweight cores. When the core and facing materials are carefully chosen and bonded firmly together, this combination provides a greater strength and rigidity with less weight than a panel consisting of a single homogeneous material would have when used in the same application (1). Sandwich panel construction is used in a variety of applications including aircraft and space vehicle structures. Walls and partitions of buildings also frequently are constructed of sandwich panels.

In order to simplify the structural analysis of these panels, a strip is chosen of some unit width and treated as a beam. Thus, this thesis refers to a sandwich beam, but the results are applicable to cylindrical bending of sandwich panels.

To illustrate the application of the derived equations, a particular beam has been chosen from previous work (2) in which natural frequencies of fixed-fixed beams were determined with no axial load applied. The values of natural frequency determined previously for this particular beam can be obtained as a special case of the current problem. The theoretical solutions to the current problem have been achieved with iteration procedures on a Burroughs 220 computer, using the Algol program language (3). By substituting the appropriate constants into the data card of the computer program used in this thesis, any particular fixed-fixed sandwich beam to which the basic assumptions are applicable may be analyzed for natural frequencies and critical loads.

CHAPTER II

LITERATURE SURVEY

Sandwich problems have commonly been solved by use of energy methods. Minimum potential energy was used by N. J. Hoff (4) to solve for deflections of a cantilever sandwich beam and for buckling of a sandwich column. Small deflection theory for sandwich construction was presented by M. E. Raville and W. R. Kimel (5) using both differential equation methods and energy methods on a sandwich panel with two sides simply supported and the other sides free. In this illustration the two approaches were found to result in identical equations. An equation for the natural frequencies of vibration of a simply supported sandwich beam was determined by Kimel, Raville, P. G. Kirmser, and M. P. Patel (6) using energy methods. When fixed-fixed sandwich beams were approached, however, the previously used energy methods failed, since functions could not be determined that met the required boundary conditions term by term and still retained the desirable property of orthogonality. To keep this orthogonality property and still meet the boundary conditions for a wider range of sandwich beam problems, Raville and E. S. Ueng (7) discussed the use of the Lagrangian multiplier method, which is based on the calculus of variations. The results were demonstrated to be more rapidly convergent when the boundary conditions of slope are satisfied by each term of the series chosen for the deflection and constraint conditions are introduced in order to satisfy the boundary conditions on deflections than when the boundary conditions of deflection are satisfied by each term of the series chosen for the deflection and constraint conditions are introduced in order to satisfy the boundary conditions on slope.

Budiansky and Hu (8) used the Lagrangian multiplier method in 1946 to solve for critical stresses in a plate clamped along all edges. Both this work and that by Raville and Ueng (7) pointed out that the use of a finite number of terms in the solution of this type of problem results in an upper limit of the eigenvalues since the equating of some terms of the infinite series equation to zero has the effect of introducing internal stiffening.

Application of the Lagrangian multiplier method to the determination of the natural frequencies of vibration of fixed-fixed sandwich beams was demonstrated by Raville, Ueng, and Ming-Min Lei (2) in 1961. Comparison of experimental results with theoretical results showed that the method quite accurately predicts the natural frequencies in this case. This solution, (2), as well as the solution of the problems of the stability of a fixed-fixed sandwich beam, can easily be obtained as a special case of the present analysis.

CHAPTER III

MATHEMATICAL ANALYSIS

The facings of the sandwich beam chosen for this thesis are assumed to be thin relative to the length of the beam. This allows the use of thin plate theory in the mathematical analysis of the beam. The facings are also assumed to be of equal thickness and homogeneous. The core is assumed to have an infinite modulus of elasticity in the direction perpendicular to the facings, and its load carrying capacity in the plane parallel to the facings is neglected. The effects of rotatory inertia are also neglected.

The coordinate system and dimensions used are shown in Figure 1. The beam is assumed to have unit width.

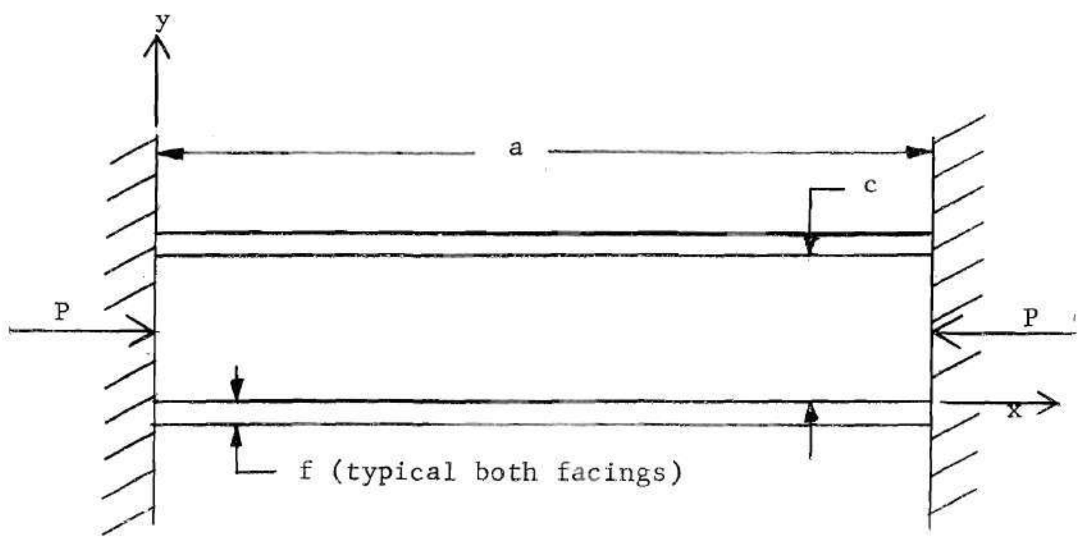

Figure 1. Sandwich Beam

In order to obtain the total elastic strain energy in the beam, expressions for the membrane strain energy in the facings, the bending strain energy in the facings, and the shearing strain energy in the core will be derived. Deflections in the x direction will be designated by u and deflections in the y direction will be designated by v.

Deflection of the core, v_c, is described by the function

$$v_c = \frac{A_0}{2} + \sum_{m=1,2,3}^{\infty} A_m \cos \frac{m\pi x}{a} \sin \omega_m t \qquad (1)$$

Shear in the core, τ_{xy}, is described by the function

$$\tau_{xy} = \sum_{m=1,2,3}^{\infty} B_m \sin \frac{m\pi x}{a} \sin \omega_m t \qquad (2)$$

Since

$$\tau_{xy} = G\left(\frac{\partial u_c}{\partial y} + \frac{\partial v_c}{\partial x}\right)$$

and

$$\frac{\partial v_c}{\partial x} = - \sum_{m=1,2,3}^{\infty} \frac{m\pi}{a} A_m \sin \frac{m\pi x}{a} \sin \omega_m t$$

then

$$\sum_{m=1,2,3}^{\infty} B_m \sin \frac{m\pi x}{a} \sin \omega_m t = G\left(\frac{\partial u_c}{\partial y} - \sum_{m=1,2,3}^{\infty} \frac{m\pi}{a} A_m \sin \frac{m\pi x}{a} \sin \omega_m t\right)$$

Therefore

$$\frac{\partial u_c}{\partial y} = \frac{\sum_{m=1,2,3}^{\infty} B_m \sin \frac{m\pi x}{a} \sin \omega_m t}{G} + \sum_{m=1,2,3}^{\infty} \frac{m\pi}{a} A_m \sin \frac{m\pi x}{a} \sin \omega_m t$$

Integrating,

$$u_c = \sum_{m=1,2,3}^{\infty} y\left(\frac{B_m}{G} + \frac{m\pi}{a} A_m\right) \sin \frac{m\pi x}{a} \sin \omega_m t + C_m \sin \frac{m\pi x}{a} \sin \omega_m t \quad (3)$$

Shear Energy in the Core

Figure 2 shows a differential element of the core. Forces in the z direction are neglected since the core is assumed not capable of carrying load in a plane parallel to the facings. The assumption of infinite modulus of elasticity of the core in the y direction eliminates the strain energy due to σ_y.

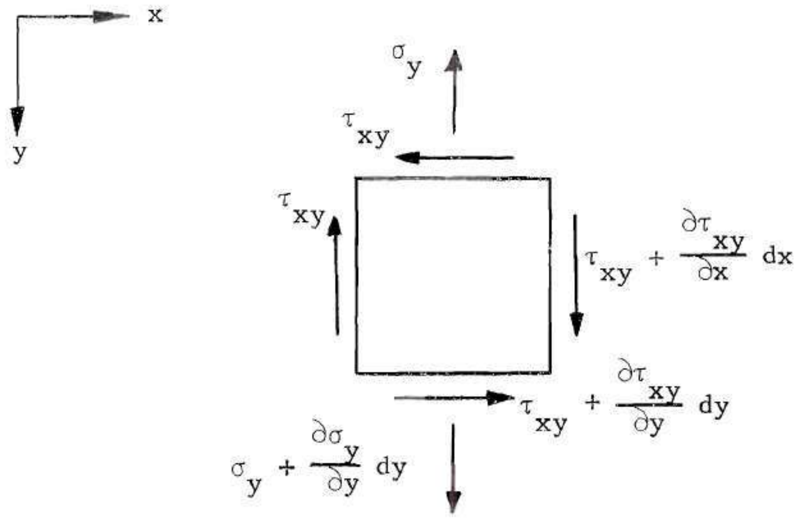

Figure 2. Differential Element of Core

The shear energy in the core is then

$$V_c = \int_0^c \int_0^a \frac{1}{2\ G} \tau_{xy}^2 \ dxdy$$

Substituting the assumed function for τ_{xy},

$$V_c = \int_0^c \int_0^a \frac{1}{2\ G} \left(\sum_{m=1,2,3}^{\infty} B_m \sin \frac{m\pi x}{a} \sin \omega_m t \right)^2 dxdy$$

or

$$V_c = \frac{1}{2G} \int_0^c \int_0^a \left[\left(\sum_{m=1,2,3}^{\infty} B_m \sin \frac{m\pi x}{a} \sin \omega_m t \right) \right.$$

$$\left. \left(\sum_{n=1,2,3}^{\infty} B_n \sin \frac{n\pi x}{a} \sin \omega_n t \right) \right] dxdy$$

Since the sine functions are orthogonal functions, i.e.,

$$\int_0^a \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{a} = 0 \quad \text{when } m \neq n$$
$$= \frac{a}{2} \quad \text{when } m = n$$

then

$$V_c = \frac{ca}{4G} \sum_{m=1,2,3}^{\infty} B_m^2 \sin^2 \omega_m t \tag{4}$$

Membrane Energy in the Facings

The deflections of the middle surface of the facings, u_f and v_f, in terms of the core deflections, u_c and v_c, are

$$u_f = (u_c)_{y=0} + \frac{f}{2} \left(\frac{\partial v_c}{\partial x} \right)_{y=0}$$

in the bottom facing (Figure 1), and

$$u_f = (u_c)_{y=c} - \frac{f}{2}\left(\frac{\partial v_c}{\partial x}\right)_{y=c}$$

in the top facing (Figure 1). The membrane strains then become

$$\epsilon_{xMb} = \frac{\partial u_f}{\partial x} = \left(\frac{\partial u_c}{\partial x} + \frac{f}{2}\,\frac{\partial^2 v_c}{\partial x^2}\right)_{y=0}$$

$$\epsilon_{xMt} = \left(\frac{\partial u_c}{\partial x} - \frac{f}{2}\,\frac{\partial^2 v_c}{\partial x^2}\right)_{y=c}$$

Displacements in the y direction are assumed to be constant through the facings and therefore

$$v_f = (v_c)_{\substack{y=0\\ y=c}}$$

The membrane strain energy for the bottom facing is

$$V_{Mb} = \int_{-f}^{0}\int_{0}^{a} \frac{E}{2(1-\nu^2)}\,\epsilon^2_{xMb}\;dxdy$$

and for the top facing,

$$V_{Mt} = \int_{c}^{c+f} \int_{0}^{a} \frac{E}{2(1-\nu^2)} \epsilon_{xMt}^2 \, dxdy$$

Considering the membrane strain energy in the bottom facing first; ϵ_{xMb} is substituted and

$$V_{Mb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_{0}^{a} \left(\frac{\partial u_c}{\partial x} + \frac{f}{2} \frac{\partial^2 v_c}{\partial x^2} \right)^2_{y=0} dxdy$$

Using the respective values of u_c and v_c and performing the partial differentiation,

$$V_{Mb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_{0}^{a} \left\{ \sum_{m=1,2,3}^{\infty} \left[\left(\frac{m\pi}{a} C_m \cos \frac{m\pi x}{a} \sin \omega_m t \right) \right. \right.$$

$$\left. \left. + \frac{f}{2} \left(- \frac{m^2\pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t \right) \right] \right\}^2 dxdy$$

or

$$V_{Mb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_{0}^{a} \left\{ \sum_{m=1,2,3}^{\infty} \left(- \frac{f}{2} \frac{m^2\pi^2}{a^2} A_m + \frac{m\pi}{a} C_m \right) \cos \frac{m\pi x}{a} \sin \omega_m t \right\}^2 dxdy$$

Integrating with respect to y,

$$V_{Mb} = \frac{Ef}{2(1-\nu^2)} \int_0^a \left\{ \left[\sum_{m=1,2,3}^{\infty} \left(-\frac{f}{2} \frac{m^2\pi^2}{a^2} A_m + \frac{m\pi}{a} C_m \right) \cos \frac{m\pi x}{a} \sin \omega_m t \right] \right.$$

$$\left. \left[\sum_{n=1,2,3}^{\infty} \left(-\frac{f}{2} \frac{n^2\pi^2}{a^2} A_n + \frac{n\pi}{a} C_n \right) \cos \frac{n\pi x}{a} \sin \omega_n t \right] \right\} dx$$

Since the cosine functions are orthogonal, i.e.,

$$\int_0^a \cos \frac{m\pi x}{a} \cos \frac{n\pi x}{a} \, dx = 0 \text{ when } m \neq n$$
$$= \frac{a}{2} \text{ when } m = n$$

it follows that

$$V_{Mb} = \frac{Efa}{4(1-\nu^2)} \sum_{m=1,2,3}^{\infty} \left[\left(-\frac{f}{2} \frac{m^2\pi^2}{a^2} A_m + \frac{m\pi}{a} C_m \right) \sin \omega_m t \right]^2 \tag{5}$$

Then, considering the membrane strain energy in the top facing and substituting ϵ_{xMt},

$$V_{Mt} = \frac{E}{2(1-\nu^2)} \int_c^{c+f} \int_0^a \left(\frac{\partial u_c}{\partial x} - \frac{f}{2} \frac{\partial^2 v_c}{\partial x^2} \right)^2_{y=c} dxdy$$

Using the respective values of u_c and v_c and performing the partial differentiation,

$$V_{Mt} = \frac{E}{2(1-\nu^2)} \int_c^{c+f} \int_0^a \left\{ \sum_{m=1,2,3}^{\infty} \left[\frac{m\pi}{a} \left(c + \frac{f}{2} \right) A_m \right. \right.$$

$$\left. \left. + B_m \frac{c}{G} + C_m \right] \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \omega_m t \right\} dxdy$$

Integrating with respect to y,

$$V_{Mt} = \frac{Ef}{2(1-\nu^2)} \int_0^a \left(\left\{ \sum_{m=1,2,3}^{\infty} \left[\frac{m\pi}{a} (c + \frac{f}{2}) A_m + B_m \frac{c}{G} \right. \right. \right.$$

$$\left. \left. + C_m \right] \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \omega_m t \right\} \left\{ \sum_{n=1,2,3}^{\infty} \left[\frac{n\pi}{a} (c + \frac{f}{2}) A_n \right. \right.$$

$$\left. \left. \left. + B_n \frac{c}{G} + C_n \right] \frac{n\pi}{a} \cos \frac{n\pi x}{a} \sin \omega_n t \right\} \right) dx$$

Again, using the orthogonality relationship,

$$V_{Mt} = \frac{Efa}{4(1-\nu^2)} \sum_{m=1,2,3}^{\infty} \left[\frac{m\pi}{a} (c + \frac{f}{2}) A_m + \frac{c}{G} B_m + C_m \right]^2 \frac{m^2\pi^2}{a^2} \sin^2 \omega_m t \quad (6)$$

Bending Energy in the Facings

The bending strain energy in the facings can be expressed as

$$V_{Bft} = \frac{E}{2(1-\nu^2)} \int_c^{c+f} \int_0^a \epsilon_{xBt}^2 \, dxdy$$

$$V_{Bfb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_0^a \epsilon_{xBb}^2 \, dxdy$$

The strain in the top facing is

$$\epsilon_{xBt} = (y-c-\frac{f}{2}) \left(\frac{\partial^2 v_c}{\partial x^2} \right)_{y=c}$$

Using the value of v_c,

$$\epsilon_{xBt} = (y-c-\frac{f}{2}) \left(\sum_{m=1,2,3}^{\infty} - \frac{m^2 \pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t \right)$$

Then

$$V_{Bft} = \frac{E}{2(1-\nu^2)} \int_c^{c+f} \int_0^a \left[(y-c-\frac{f}{2}) \sum_{m=1,2,3}^{\infty} - \frac{m^2 \pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t \right]^2 dxdy$$

or

$$V_{Bft} = \frac{E}{2(1-\nu^2)} \int_c^{c+f} \int_0^a \left[(y-c-\frac{f}{2}) \sum_{m=1,2,3}^{\infty} - \frac{m^2\pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t \right]$$

$$\left[(y-c-\frac{f}{2}) \sum_{n=1,2,3}^{\infty} - \frac{n^2\pi^2}{a^2} A_n \cos \frac{n\pi x}{a} \sin \omega_n t \right] dxdy$$

Again, since $\cos \frac{m\pi x}{a}$ and $\cos \frac{n\pi x}{a}$ are orthogonal,

$$V_{Bft} = \frac{E}{2(1-\nu^2)} \frac{a}{2} \sum_{m=1,2,3}^{\infty} \frac{m^4\pi^4}{a^4} A_m^2 \sin^2 \omega_m t \int_c^{c+f} (y-c-\frac{f}{2})^2 \, dy$$

Integrating with respect to y,

$$V_{Bft} = \frac{E}{2(1-\nu^2)} \frac{a}{2} \frac{f^3}{12} \sum_{m=1,2,3}^{\infty} \frac{m^4\pi^4}{a^4} A_m^2 \sin^2 \omega_m t \tag{7}$$

The strain in the bottom facing is

$$\epsilon_{xBb} = (y+\frac{f}{2}) \left(\frac{\partial^2 v_c}{\partial x^2} \right)_{y=0}$$

Using the value of v_c,

$$\epsilon_{xBb} = \left(y + \frac{f}{2}\right) \sum_{m=1,2,3}^{\infty} - \frac{m^2 \pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t$$

Then

$$V_{Bfb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_{0}^{a} \left[\left(y + \frac{f}{2}\right) \sum_{m=1,2,3}^{\infty} - \frac{m^2 \pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t\right]^2 dxdy$$

or

$$V_{Bfb} = \frac{E}{2(1-\nu^2)} \int_{-f}^{0} \int_{0}^{a} \left[\left(y + \frac{f}{2}\right) \sum_{m=1,2,3}^{\infty} - \frac{m^2 \pi^2}{a^2} A_m \cos \frac{m\pi x}{a} \sin \omega_m t\right]$$

$$\left[\left(y + \frac{f}{2}\right) \sum_{n=1,2,3}^{\infty} - \frac{n^2 \pi^2}{a^2} A_n \cos \frac{n\pi x}{a} \sin \omega_n t\right] dxdy$$

As before, since $\cos \frac{m\pi x}{a}$ and $\cos \frac{n\pi x}{a}$ are orthogonal,

$$V_{Bfb} = \frac{E}{2(1-\nu^2)} \frac{a}{2} \sum_{m=1,2,3}^{\infty} \frac{m^4 \pi^4}{a^4} A_m^2 \sin^2 \omega_m t \int_{-f}^{0} \left(y + \frac{f}{2}\right)^2 dy$$

Integrating,

$$V_{Bfb} = \frac{E}{2(1-\nu^2)} \frac{a}{2} \frac{f^3}{12} \sum_{m=1,2,3}^{\infty} \frac{m^4 \pi^4}{a^4} A_m^2 \sin^2 \omega_m t \tag{8}$$

Total Elastic Strain Energy

The total elastic strain energy is the sum of the shear energy in the core, the membrane energy in the facings and the bending energy in the facings. The total elastic strain energy is then

$$V = V_c + V_{Mb} + V_{Mt} + V_{Bft} + V_{Bfb}$$

or

$$V = \frac{ca}{4G} \sum_{m=1,2,3}^{\infty} B_m^2 \sin^2 \omega_m t + \frac{Efa}{4(1-\nu^2)} \sum_{m=1,2,3}^{\infty} \left[\left(- \frac{f}{2} \frac{m^2 \pi^2}{a^2} A_m + \frac{m\pi}{a} C_m \right) \sin \omega_m t \right]^2$$

$$+ \frac{Efa}{4(1-\nu^2)} \sum_{m=1,2,3}^{\infty} \left[\frac{m\pi}{a} (c + \frac{f}{2}) A_m + \frac{c}{G} B_m + C_m \right]^2 \frac{m^2 \pi^2}{a^2} \sin^2 \omega_m t$$

$$+ \frac{E}{(1-\nu^2)} \frac{a}{2} \frac{f^3}{12} \sum_{m=1,2,3}^{\infty} \frac{m^4 \pi^4}{a^4} A_m^2 \sin^2 \omega_m t$$

Energy Lost by the Applied Load, P

$$T_P = \frac{P}{2}\int_0^a \left(\frac{\partial v_c}{\partial x}\right)^2 dx$$

Using the value of v_c,

$$T_P = \frac{P}{2}\int_0^a \left(-\sum_{m=1,2,3}^{\infty} \frac{m\pi}{a} A_m \sin\frac{m\pi x}{a} \sin \omega_m t\right)^2 dx$$

or

$$T_P = \frac{P}{2}\int_0^a \left(\sum_{m=1,2,3}^{\infty} \frac{m\pi}{a} A_m \sin\frac{m\pi x}{a} \sin \omega_m t\right)\left(\sum_{n=1,2,3}^{\infty} \frac{n\pi}{a} A_n \sin\frac{n\pi x}{a} \sin \omega_n t\right) dx$$

Sin $\frac{m\pi x}{a}$ and sin $\frac{n\pi x}{a}$ are orthogonal functions, therefore

$$T_P = \frac{P}{2}\,\frac{a}{2}\sum_{m=1,2,3}^{\infty} \frac{m^2\pi^2}{a^2} A_m^2 \sin^2 \omega_m t \qquad (9)$$

Kinetic Energy of the Vibrating Beam

$$T_K = \frac{\rho}{2}\int_0^a \left(\frac{\partial v_c}{\partial t}\right)^2 dx$$

Using the value of v_c,

$$T_K = \frac{\rho}{2}\int_0^a \left[\omega_m \left(\frac{A_0}{2} + \sum_{m=1,2,3}^{\infty} A_m \cos\frac{m\pi x}{a} \cos\omega_m t\right)\right]^2 dx$$

or

$$T_K = \frac{\rho}{2}\int_0^a \left\{\left[\omega_m \left(\frac{A_0}{2} + \sum_{m=1,2,3}^{\infty} A_m \cos\frac{m\pi x}{a} \cos\omega_m t\right)\right]\right.$$

$$\left.\left[\omega_n \left(\frac{A_0}{2} + \sum_{n=1,2,3}^{\infty} A_n \cos\frac{n\pi x}{a} \cos\omega_n t\right)\right]\right\} dx$$

Since $\cos\frac{m\pi x}{a}$ and $\cos\frac{n\pi x}{a}$ are orthogonal functions and

$$\int_0^a \cos\frac{m\pi x}{a}\, dx = 0$$

$$T_K = \frac{\rho}{2}\frac{a}{2}\left(\frac{A_0^2}{2} + \sum_{m=1,2,3}^{\infty} \omega_m^2 A_m^2 \cos^2 \omega_m t\right) \tag{10}$$

Constraints

The assumed deflection function, v_c, meets the boundary condition for slope at x=0 and x=a, since

$$\frac{\partial v_c}{\partial x} = \sum_{m=1,2,3}^{\infty} \left(-\frac{m\pi}{a} A_m \sin \frac{m\pi x}{a} \sin \omega_m t\right)$$

is the corresponding equation for slope and it becomes zero at x=0 and x=a.

However, the assumed deflection function does not satisfy the required boundary conditions that the deflections become zero at x=0 and x=a. In order to satisfy these boundary conditions the following constraint conditions must be imposed;

$$\frac{A_0}{2} + \sum_{m=1,2,3}^{\infty} A_m = 0$$

and

$$\frac{A_0}{2} + \sum_{m=1,2,3}^{\infty} (-1)^m A_m = 0$$

When even and odd terms are considered separately, both constraints reduce to a similar form

$$\sum_{m=1,3,5}^{\infty} A_m = 0 \tag{11}$$

and

$$\frac{A_0}{2} + \sum_{m=2,4,6}^{\infty} A_m = 0 \tag{12}$$

The constraints are multiplied by the Lagrangian multipliers, λ_1 and λ_2 respectively, and added to the total energy expression. The modified energy expression then is

$$U = V - T_P - T_K + \lambda_1 \sum_{m=1,3,5}^{\infty} A_m + \lambda_2 \left(\frac{A_0}{2} + \sum_{m=2,4,6}^{\infty} A_m \right)$$

The system is considered conservative so that

$$U = V_{max} - T_{Pmax} - T_{Kmax} + \lambda_1 \sum_{m=1,3,5}^{\infty} A_m + \lambda_2 \left(\frac{A_0}{2} + \sum_{m=2,4,6}^{\infty} A_m \right)$$

must be stationary with respect to the configuration parameters A_0, A_m, B_m, and C_m.

For the maximum values of V, T_P, and T_K, which are V_{max}, T_{Pmax}, and T_{Kmax}, $\sin^2 \omega_m t$ and $\cos^2 \omega_m t$ must be equal to unity. Then,

$$U = \sum_{m=1,2,3}^{\infty} \left\{ \frac{ca}{4G} B_m^2 + \frac{Efa}{4(1-\nu^2)} \left(- \frac{f}{2} \frac{m^2\pi^2}{a^2} A_m + \frac{m\pi}{a} C_m \right)^2 \right. \tag{13}$$

$$+ \frac{Efa}{4(1-\nu^2)} \frac{m^2\pi^2}{a^2} \left[\frac{m\pi}{a} (c + \frac{f}{2}) A_m + \frac{c}{G} B_m + C_m \right]^2$$

$$+ \frac{E}{(1-\nu^2)} \frac{a}{2} \frac{f^3}{12} \frac{m^4\pi^4}{a^4} A_m^2 - \frac{P}{2} \frac{a}{2} \frac{m^2\pi^2}{a^2} A_m^2$$

$$\left. - \frac{\rho}{2} \omega_m^2 \frac{a}{2} \left(\frac{A_0^2}{2} + A_m^2 \right) \right\} + \lambda_1 \sum_{m=1,3,5}^{\infty} A_m + \lambda_2 \left(\frac{A_0}{2} + \sum_{m=2,4,6}^{\infty} A_m \right)$$

Then, setting the partial derivative of U with respect to each of the configuration parameters A_0, A_i, B_i, and C_i equal to zero,

$$\frac{\partial U}{\partial A_0} = 0 = \frac{\lambda_2}{2} - \frac{\rho}{2} \omega^2 \frac{a}{2} A_0$$

Therefore,

$$\frac{A_0}{2} = - \frac{\lambda_2}{\rho\omega^2 a} \tag{14}$$

and

$$\frac{\partial U}{\partial A_i} = 0 = \left\{ \frac{Efa}{2(1-\nu^2)} \frac{i^4\pi^4}{a^4} \left[\frac{5f^2}{12} + (c + \frac{f}{2})^2 \right] \right. \tag{15}$$

$$\left. - \frac{Pa}{2} \frac{i^2\pi^2}{a^2} - \rho\omega^2 \frac{a}{2} \right\} A_i + \frac{Efa}{2(1-\nu^2)} \frac{i^3\pi^3}{a^3} (c + \frac{f}{2}) \frac{c}{G} B_i$$

$$+ \frac{Efa}{2(1-\nu^2)} \frac{i^3\pi^3}{a^3} c^3 C_i + \lambda_1 + \lambda_2$$

$$\frac{\partial U}{\partial B_i} = 0 = \frac{i\pi}{a} (c + \frac{f}{2}) A_i + \left(\frac{a^2(1-\nu^2)}{Efi^2\pi^2} + \frac{c}{G} \right) B_i + C_i \tag{16}$$

$$\frac{\partial U}{\partial C_i} = 0 = \frac{i\pi c}{a} A_i + \frac{c}{G} B_i + 2C_i \tag{17}$$

These four equations can then be solved for the configuration parameters:

$$A_i = \frac{-\lambda_1 - \lambda_2}{\frac{i^4\pi^4}{a^4}\,\frac{a}{2}\,\frac{E}{(1-\nu^2)}\left(I_F + \frac{I_T}{i^2S + 1}\right) - P\,\frac{a}{2}\,\frac{i^2\pi^2}{a^2} - \rho\,\frac{a}{2}\,\omega^2} \tag{18}$$

$$B_i = -\frac{i\pi G}{a}\left(1 + \frac{i^2Sf - c}{c(i^2S + 1)}\right) A_i$$

$$C_i = \frac{i\pi}{2a}\left(\frac{i^2Sf - c}{i^2S + 1}\right) A_i$$

Where

$$S = \frac{\pi^2 cf}{2a^2}\,\frac{E}{G(1-\nu^2)}$$

$$I_F = \frac{f^3}{6}$$

$$I_T = \frac{f}{2}\,(c + f)^2$$

Then substituting A_i into the constraint condition for odd terms (equation 11),

$$\sum_{m=1,3,5}^{\infty} \frac{-\lambda_1}{\frac{m^4\pi^4}{a^4}\frac{a}{2}\frac{E}{(1-\nu^2)}\left(I_F + \frac{I_T}{m^2S+1}\right) - P\frac{a}{2}\frac{m^2\pi^2}{a^2} - \rho\,\omega^2\frac{a}{2}} = 0$$

Assuming $-\lambda_1$ is not zero,

$$\sum_{m=1,3,5}^{\infty} \frac{1}{\frac{m^4\pi^4}{a^4}\frac{E}{(1-\nu^2)}\left(I_F + \frac{I_T}{m^2S+1}\right) - P\frac{m^2\pi^2}{a^2} - \rho\,\omega^2} = 0 \qquad (19)$$

Substituting $\frac{A_0}{2}$ and A_i into the constraint condition for even terms (equation 12),

$$\frac{\lambda_2}{\rho\omega^2 a} + \sum_{m=2,4,6}^{\infty} \frac{-\lambda_2}{\frac{m^4\pi^4}{a^4}\frac{a}{2}\frac{E}{(1-\nu^2)}\left(I_F + \frac{I_T}{m^2S+1}\right) - P\frac{a}{2}\frac{m^2\pi^2}{a^2} - \rho\,\omega^2\frac{a}{2}} = 0$$

Assuming $-\lambda_2$ is not zero,

$$\frac{-1}{2\rho\omega^2} + \sum_{m=2,4,6}^{\infty} \frac{1}{\frac{m^4\pi^4}{a^4} \frac{E}{(1-\nu^2)} \left(I_F + \frac{I_T}{m^2 S + 1} \right) - P \frac{m^2\pi^2}{a^2} - \rho\omega^2} = 0 \quad (20)$$

Equations (19) and (20) comprise the formal solution to the title problem.

CHAPTER IV

APPLICATION

The equations derived to describe the axial load and frequency relationship of the fixed-fixed sandwich beam are in the form of infinite series. A finite number of terms must be chosen to achieve a solution of the equations, and because of the limited scope of the thesis, the number of terms in each series was limited to six. This was considered to be sufficiently accurate for the first two modes in the current work, based on results presented by Raville, Ueng, and Lei (2). They state that, based on trials with 26 terms on an equation for vibration only, accuracy of the fundamental frequency when only three terms are used is within one and one-half percent.

The six-term left hand side of the equation does not lend itself to algebraic solution. The values of load and frequency which are solutions to the equations are best determined with the use of a digital computer.

In adapting the problem to the computer, an iteration procedure was used to find values of load and frequency which satisfied equations (19) and (20). In each case, a load of fixed value was chosen and the natural frequency which caused the equation to approach zero with the fixed load was found. The load was then varied by an arbitrary amount and the procedure repeated.

While writing the program for the iteration procedures on the computer, it was found that care had to be used to assure that indeterminate points were not written out by the computer as solutions. Each term of the series became indeterminate when its denominator approached zero; at any particular value of axial load, an indeterminate point occurred once for each term used. As the frequency was increased, the indeterminate point occurred in each term consecutively from left to right. A point occurred between each of the indeterminate points at which the value of the equation became zero. This point is a solution of the equation. All variables except frequency in the terms of the series were fixed when an iteration pass was made. Frequency has a negative sign in all terms, so that increasing frequency always decreased the value of the terms which were positive. Thus, the denominators of the terms which caused the indeterminate points always crossed zero from the positive to the negative side as frequency was increased. Since the value of the term was inordinately large as its denominator approached zero, it dominated the series and caused the sign of the series to cross zero from the positive to the negative side at the indeterminate points. Thus the indeterminate points are characterized by the value of the series going from positive to negative as frequency is increased.

As frequency is increased still further, the value of the particular dominating term begins to decrease as its denominator increases negatively so that the term to the right, being positive, balances the negative terms to the left and causes the value of the series to cross zero. Thus, the points at which the value of the equation becomes zero are characterized by the fact that the value of the series crosses zero from the negative to the positive side. These zero points are the solutions of the equation.

The foregoing characteristics are taken into account in writing the computer program to assure that the values of frequency obtained are solutions to the equation. As a check, the value of the series is printed out. Since the value of the series approaches zero at the solution, the value of the series which is printed out is always small.

A beam having the same dimensions and physical properties as one of those used by Raville, Ueng, and Lei (2) was chosen for application. The beam chosen is one with a 1.000 inch aluminum core and 0.016 inch aluminum facings. It has a Poisson's ratio of 0.33, a modulus of elasticity of the facings of 10.6×10^6 psi, a modulus of rigidity in the core of 8×10^3 psi, and a mass density of the composite beam per unit length and width of 1.38745×10^{-5} lb.sec.^{2}/in.^{3}.

CHAPTER V

DISCUSSION

The equations derived for the combined vibration and stability of the sandwich beam are similar to those derived by Raville, Ueng, and Lei (2) for vibration alone and to equations the author has derived for stability alone. The vibration problem and the stability problem are each a special case of the combined problem. The combined problem becomes the same as the vibration problem when the axial load in the combined problem is made zero, and the combined problem approaches the stability problem when the natural frequency in the combined problem approaches zero.

The data acquired from equations (19) and (20) for the beam described in Chapter IV is plotted in Figure 3 to show the interaction of axial load and natural frequency of the beam. The slope of the interaction curves is relatively low at lesser values of axial load so that little change in the natural frequency occurs with change in axial load. However, as the axial load approaches a critical value, the slope becomes greater, thus indicating that natural frequencies change considerably in this range of axial load. This shape of the curve is caused by the fact that the frequency is squared in the denominator of each term, while the axial load occurs with a unit exponent. Since both axial load and frequency occur with negative signs in each of the terms of the series, while the inertia portion of each term remains constant, a decrease in axial load causes a corresponding change in the squared frequency and its

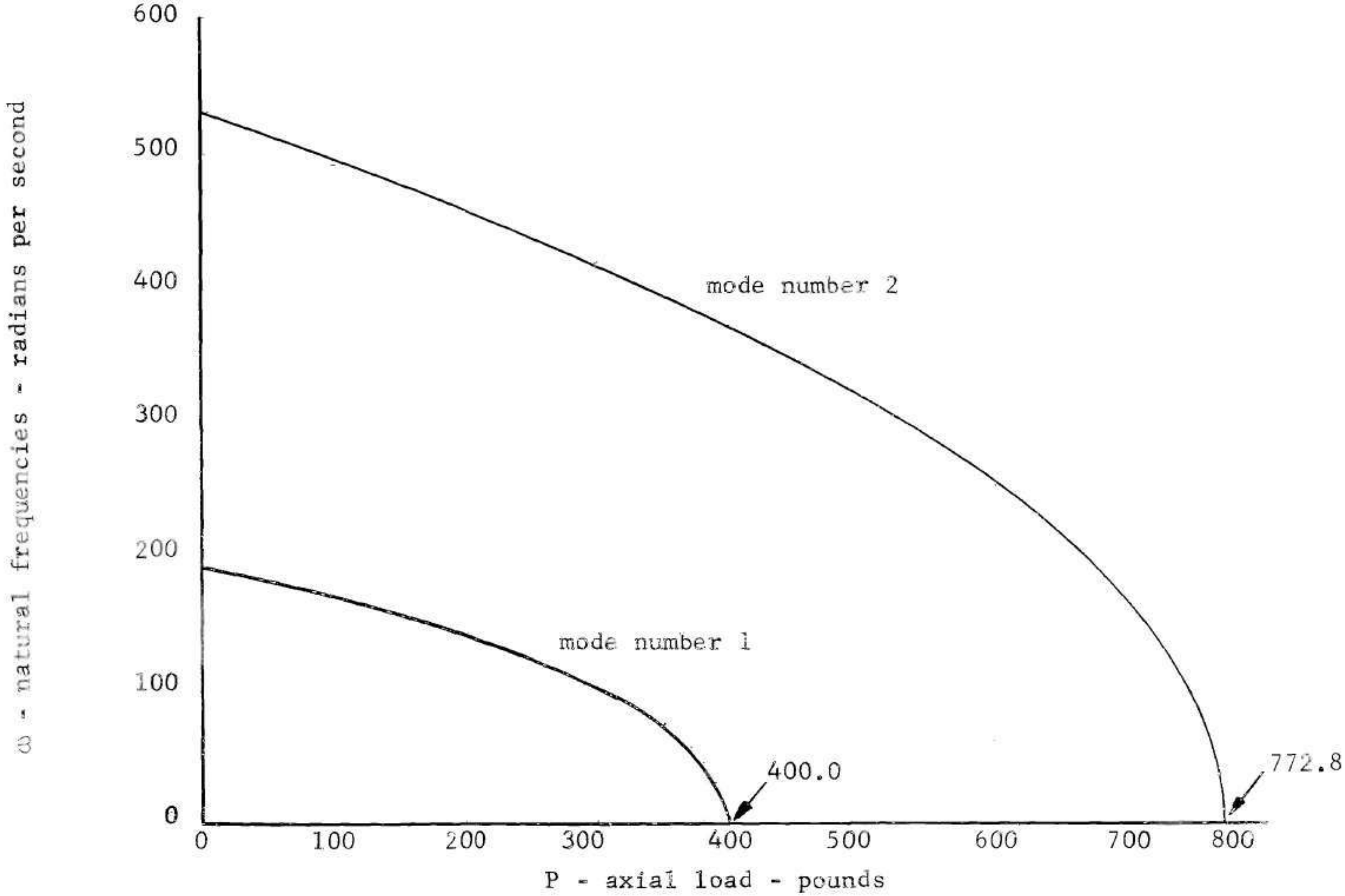

Figure 3. Natural Frequencies vs Axial Load

For a 96 inch Beam - First Two Modes

coefficient. As a result the interaction curves in Figure 3 are nonlinear.

Values of load and natural frequency were also obtained for the third, fourth, and fifth modes for axial loads up to 1000 pounds. These values, however, are not considered of sufficient accuracy for detailed consideration and are included in Figure 4 for information only.

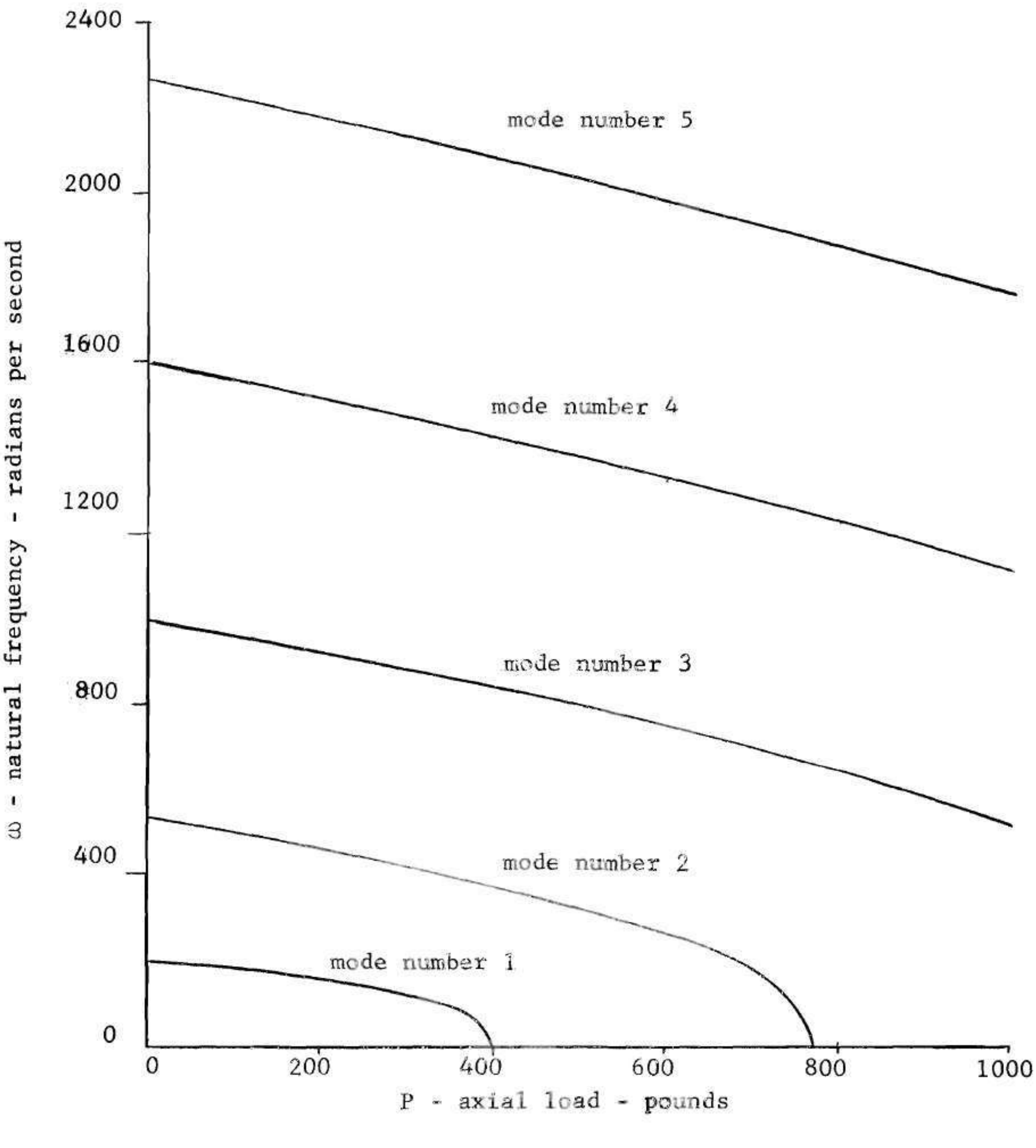

Figure 4. Natural Frequency vs Axial Load

For A 96 inch Beam - Modes 1 Through 5

CHAPTER VI

CONCLUSIONS

The equations derived to determine axial load versus natural frequency for the fixed-fixed sandwich beam are similar to those derived for stability without vibration and to those derived for natural frequencies of vibration with no axial load applied.

At values of axial load approaching the critical, a small change in the load causes the natural frequency of the beam to change extensively; as the axial load decreases, however, the natural frequency changes less rapidly with a change in axial load.

The program written to acquire the data plotted in Figure 3 can be used to acquire similar data on any given fixed-fixed sandwich beam with facings of equal thickness by replacing the pertinent constants on the data card at the end of the program. With minor changes in the program, the number of terms in the series and the number of modes can be increased.

CHAPTER VII

RECOMMENDATIONS

Experimental verification of the results obtained in this study would be desirable. In addition, practical limits of length, core thickness, and facing thickness need to be determined both analytically and experimentally. Analysis and tests of a wide spectrum of dimensions and material variations of the fixed-fixed sandwich beam under load and vibration would add to the available structural design information on sandwich construction.

It is stated in a previous work (2) that, at the lower frequencies, homogeneous beam theory provides a satisfactory approximation of the natural frequencies of a sandwich beam. The accuracy of this approximation when applied to the combined case of vibration and axial loads on a fixed-fixed beam could be determined by obtaining the natural frequency versus axial load relationship for a homogeneous beam with the same length and with a moment of inertia equal to the transfer term in the moment of inertia of the facings of the one studied in this thesis and comparing the results to those obtained in this thesis. Comparison of these results would, in addition, give indication of any approximation that might be used for the critical buckling loads.

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